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A model of turbulent mixing across a density interface including the effect of rotation

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A new theoretical approach is presented, which determines the entrainment laws with and without background rotation in a two-layer stratified fluid with one layer stirred by turbulent motions. The model gives the entrainment coefficient E as a function of the local turbulent Richardson number $Ri = g'l/u^2$, the Rossby number $Ro = u/fl$ and the Péclet number $Pe = ul/\kappa$. The following entrainment laws are obtained: (i) without rotation and for a high Péclet number: $E \propto Ri^{-\frac{3}{2}}$; (ii) without rotation and for a low to moderate Péclet number: $E \propto Pe^{-\frac{1}{3}} Ri^{-1}$; (iii) with rotation and for a high Péclet number: $E \propto Ro\, Ri^{-1}$. These entrainment laws are consistent with experiments for the three different cases. The model relies to a great extent on the spectral distribution of interface oscillations measured in experiments. Comparison is made with experiments and with earlier models of entrainment.

1. Introduction

Theoretical aspects of mixing at a density interface between two superimposed miscible fluids when one layer is at rest and the other is stirred by turbulent motions are considered. The density stratification in the interface is assumed to be strong, meaning that the turbulent Richardson number

$$Ri = \frac{g'l}{u^2} = \frac{\Delta\rho g l}{\rho u^2}, \tag{1}$$

is high ($Ri \gg 1$). The reduced gravity is g', and u and l designate respectively the horizontal component of turbulent velocity and the turbulent lengthscale at the interface. In what follows the stirred layer is the lower layer.

There have been a large number of experimental investigations of turbulent entrainment across density interfaces in a non-rotating fluid subjected to turbulence when the Péclet number is high (Turner 1973; Thompson & Turner 1975; Hopfinger & Toly 1976; E & Hopfinger 1986; Linden 1973; Long 1978; Fernando & Long 1985). Except for the work of Long (1978) and Fernando & Long (1985), all these studies agree on the following dependence of the entrainment rate on the Richardson number:

$$E = \frac{u_e}{u} = K Ri^{-\frac{3}{2}}, \tag{2}$$

where u_e is the mean displacement velocity of the interface ('entrainment velocity'). The constant K varies slightly from one author to another and from one experiment to another. It is in general of $O(1)$. A theoretical justification of the experimental law (2) was proposed by Linden (1973). His model, which is supported by a simple experiment, considers the mixing caused by turbulent eddies, modelled by a vortex ring impinging perpendicularly on the interface. The distortion of the interface forces the eddy to recoil into the turbulent layer, carrying lighter fluid inside the stirred fluid during the recoil. A dimensional analysis of this mechanism led Linden to the $Ri^{-\frac{3}{2}}$ law.

The present paper considers another entrainment mechanism. Mixing is assumed to be caused by small turbulent eddies of Richardson number

$$\frac{g(\partial\rho/\partial z)}{\rho(\partial u/\partial z)^2}$$

of the order of one or less. By mixing the density profile, these eddies convert kinetic energy into potential energy, at a rate given by their own timescale and at a scale given by their own lengthscale. The entrainment law is obtained by integration over all eddies of Richardson number below 1.

This analysis was developed simultaneously with an experimental study (Fleury *et al.* 1991), aimed at understanding the effect of rotation on turbulent entrainment. The study was motivated by geophysical applications. An entrainment law was obtained experimentally in the form

$$E = 0.5 Ro\, Ri^{-1}, \tag{3}$$

$Ro = u/fl$ being the Rossby number based on the turbulent velocity u and integral lengthscale l. By developing a new approach to understanding the effect of rotation on mixing, it appeared that this new theory could easily be generalized to predict the entrainment behaviour obtained in non-rotating experiments for high and moderate Péclet number $Pe = ul/\kappa$.

In what follows emphasis will be placed on knowledge of the turbulent kinetic energy spectrum near the interface. Important theoretical attempts are being made at present by Carruthers and Hunt to estimate these quantities. In an earlier study Carruthers & Hunt (1986) considered density profiles with a jump in Brunt–Väisälä frequency at the interface, but without a density jump. Work is currently in progress by these authors on density profiles presenting a density jump across an interface of thickness e. Their theoretical analysis focuses on determination of the velocity field near the interface. In the present paper the interest is in mixing, for which the structure of turbulence needs to be known. Reference is made to the experimental studies by Hannoun, Fernando & List (1988) and Hannoun & List (1988) (without rotation) and by Fleury *et al.* (1990) (with rotation). Their results concerning the behaviour of the kinetic energy distribution at the interface are used. These serve to establish the entrainment law as well as to justify certain assumptions of the theory.

The basic principles of the theory are presented in §2. The results are then applied to mixing in a turbulent flow with no rotation (§3). The effect of rotation is considered in the model in §4. Finally §5 discusses and compares this theory to earlier theoretical models by Linden (1973), Long (1978) and Phillips (1977).

2. Theoretical foundations

We consider, as depicted in figure 1, a stratified fluid formed by two layers with densities ρ and $\rho + \Delta\rho$, separated by an interface of thickness e. The lower layer is subjected to turbulent stirring, produced for example by an oscillating grid. This turbulence is assumed to be statistically homogeneous. The turbulent velocity is denoted u and the turbulent lengthscale is l. The depth of the lower layer is h. When

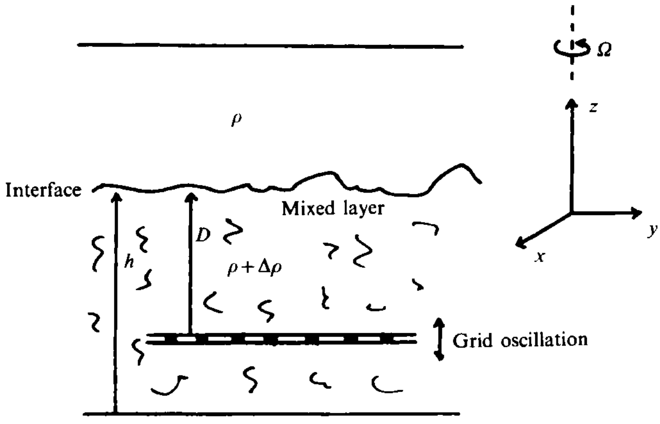


FIGURE 1. Schematic diagram of the flow, and notation.

effects of rotation are considered in §4 of this paper, the rotation is around the z-axis parallel to gravity $\boldsymbol{g}$.

Since the Richardson $Ri = g'l/u^2$ is high, the fluid mixed at the interface is entrained into and mixed throughout the turbulent lower layer, whereas the density in the upper layer remains constant. Mixing at the interface induces a slow displacement of the mean position of the interface. The entrainment velocity, $u_e = \mathrm{d}h/\mathrm{d}t$, and the entrainment coefficient is defined by $E = u_e/u$.

As previously considered by various authors (Phillips 1977; Gibson 1981, among others), the physical mechanism producing the mixing of the density profile is assumed to be an instability of Kelvin–Helmholtz type. The main consequence of a high Richardson number is that eddies of the order of the integral lengthscale l are unable to mix the density profile. The general criterion for the Kelvin–Helmholtz instability is in the form

$$\frac{g(\partial\rho/\partial z)}{\rho(\partial u/\partial z)^2} = \frac{g'}{e(\partial u/\partial z)^2} \leqslant 0.25. \tag{4}$$

Estimating this quantity for the large-scale turbulent eddies, $\partial u/\partial z \approx u/l$ implies

$$\frac{g'}{e(\partial u/\partial z)^2} \approx \frac{l}{e} Ri \gg 1$$

because the ratio l/e of the integral lengthscale and the interface thickness is greater than one for all experiments referred to in the present paper. We therefore verify that criterion (4) for instability is not satisfied for eddies of size l. Nevertheless, the turbulent flow contains various scales of eddies, and the smaller ones may satisfy criterion (4) and mix the fluid. In order to determine the range of eddies that can mix the fluid, the local characteristic frequency $\partial u/\partial z$ has to be estimated for each scale of eddies and compared using (4) to the Brunt–Väisälä frequency $(g'/e)^{\frac{1}{2}}$. The relative contribution to the mixing of each scale of motion is estimated by using a spectral description of the turbulent flow. $E_{\mathrm{K}}(k)$ designates the kinetic energy spectral density (k is the wavenumber modulus). Two particular frequencies characterize a turbulent eddy. The first one is the turnover frequency of this eddy when the observer moves with the large-scale motions. We therefore call this frequency the 'Lagrangian frequency':

$$\omega_{\mathrm{L}} \approx (E_{\mathrm{K}}(k)\,k)^{\frac{1}{2}} k. \tag{5}$$

The second frequency, called 'Eulerian frequency', is that measured by a fixed observer seeing an eddy with scale $1/k$ translated by the large-scale motions:

$$\omega_E \approx uk. \tag{6}$$

The Eulerian frequency is more representative of the shear produced at the interface at scale $1/k$, because the interface is sharp and motions are significantly weaker above the interface than below. Introducing the Eulerian frequency in criterion (4), the size $1/k$ of eddies contributing to mixing is given by the range

$$kl \geqslant \left(\frac{Ri\,l}{e}\right)^{\frac{1}{2}}. \tag{7}$$

When mixing proceeds, the mean position of the interface moves slowly with the entrainment velocity u_e. The rate of increase of the total potential energy per unit volume is

$$\frac{dE_p}{dt} \approx \Delta\rho g h u_e \tag{8}$$

because the interface thickness is assumed to be small compared to the depth of the turbulent layer ($e/h \ll 1$). Turbulent eddies, when mixing the fluid, transform kinetic energy into potential energy, so that the rate of potential energy increase may be expected to be proportional to the kinetic energy per unit mass available just below the interface. The total mixing is the sum of contributions from those turbulent eddies contributing to mixing as determined by (7). From a dimensional argument, it may be deduced that

$$\frac{dE_p}{dt} = \rho \int_I E_k(k)\,\omega(k)\,\lambda(k)\,dk, \tag{9}$$

where $\omega(k)$ and $\lambda(k)$ designate respectively the characteristic frequency and lengthscale at which the turbulent eddy with wavenumber k mixes the density profile. The domain of integration I is given by (7).

It is worth mentioning that the dimensional argument used to establish (9) is quite similar to the procedure followed by Linden (1973). In both cases the rate of increase is assumed to be proportional to the available turbulent energy multiplied by a characteristic lengthscale and divided by a characteristic timescale. However, the mechanisms are completely different. Here the mixing is caused by Kelvin–Helmholtz instabilities and is therefore a small-scale process, whereas Linden associates the mixing directly with the dynamics of large-scale eddies.

Returning to (9), a mixing-length argument provides a simple estimate of the lengthscale $\lambda(k)$:

$$\lambda(k) \approx 1/k, \tag{10}$$

which means that an eddy in contact with the interface cannot mix the fluid on a scale larger than its own size.

It is not, *a priori*, obvious whether the Eulerian frequency ω_E or the Lagrangian frequency ω_L gives the relevant timescale of mixing by an eddy with wavenumber k. There are arguments in favour of choosing the Eulerian timescale, since this frequency was previously chosen to establish criterion (7) determining the size of eddies contributing to the mixing, and because the Eulerian frequency is more appropriate for estimating the real shear at the interface. In linear models of the Kelvin–Helmholtz instability, the Eulerian frequency actually scales the exponential growth of the instability. The Lagrangian frequency $\omega(k) = \omega_L(k)$ (equation (5)) is nevertheless chosen. This choice is motivated by the assumption that exponential growth represents only a short period of time at the beginning of the mixing process. After some time, the entrained fluid is transported by the large-scale eddies, and the mixing, which is produced by the smaller eddies, continues with the Lagrangian time scale $1/\omega_L$.

The growth of the potential energy ((8) and (9)) then becomes

$$\frac{dE_p}{dt} = \Delta\rho g h u_e = \rho \int_{kl > (Ri\, l/e)^{\frac{1}{2}}} E_k(k)\,(E_k(k)\,k)^{\frac{1}{2}}\,dk, \tag{11}$$

after using (5), (7) and (10). The turbulent kinetic energy spectrum has now to be determined, and this is done in the next sections when the cases without and with rotation are considered successively. It is worth mentioning here that turbulence is almost isotropic at the small scales at which mixing occurs. In non-rotating conditions this property is demonstrated by measurements made by Hannoun *et al.* (1988), as we shall see later on. This fact removes any ambiguity surrounding the definition of the turbulent kinetic energy spectrum $E_k(k)$ used in this paper.

3. Application to non-rotating turbulent flows

In this section, the entrainment law is determined for two asymptotic cases, depending on whether molecular diffusion is efficient (small or moderate Péclet number) or negligible ($Pe \gg 1$). As shown by (11), the dependence of the ratio of the interface thickness to the integral lengthscale e/l on the various non-dimensional numbers governing the flow must be determined to deduce the entrainment law.

Considering first the case of very high-Péclet-number flows, experiments (Crapper & Linden 1974; E & Hopfinger 1986; Hannoun & List 1988) have shown that e/l does not depend on the Richardson number when this number is sufficiently large ($Ri > 50$). The asymptotic value of e/l varies considerably from one experiment to another. This value, obtained from instantaneous density profiles measured using a conductivity probe, is relatively large in the experiments by Crapper & Linden ($e/l \approx 1.5$) and E & Hopfinger ($e/l \approx 0.24$). The more recent technique developed by Hannoun & List (1988), using laser-induced fluorescence to measure instantaneous concentration profiles of Rhodamine dye, gave a much smaller asymptotic interface thickness ($e/l \approx 0.04$ for $Ri > 50$). The latter authors mentioned that this ratio may even be overmeasured because of a lack of resolution at high Ri. The discrepancy in the asymptotic value of e/l is not understood. Anyway, it is sufficient in the framework of the present model to assume that e/l does not depend on Ri for high Ri,

$$e/l \approx \text{const}, \tag{12}$$

a result verified by all experiments. However, a dependence of e/l on the Péclet number cannot be excluded as considered by Hannoun & List (1988).

When the Péclet number is low or moderate, molecular diffusion plays a key role at the interface by determining its thickness. Molecular diffusion tends to thicken the interface according to

$$e \approx (\kappa t)^{\frac{1}{2}}, \tag{13}$$

which implies

$$\frac{de}{dt} \approx \frac{\kappa}{e}. \tag{14}$$

This typical velocity of diffusion of the interface decreases as the interface thickness grows. On the other hand, entrainment tends to decrease the interface thickness. We consider here the case when the interfacial layer is in equilibrium, that is when the increase in the interface thickness due to diffusion is balanced by the decrease due to the entrainment. This balance between entrainment and diffusion therefore leads to

$$E = \frac{u_e}{u} \approx \frac{\kappa}{eu} \approx \frac{l}{e} Pe^{-1}. \tag{15}$$

Less experimental data on the interface thickness are available when the Péclet number is small or moderate than when it is very high. Some comparison with the results obtained by Crapper & Linden (1974) is made at the end of the present section, after the entrainment law is established.

The entrainment law is now determined from (11). The closure of this relationship follows from (12) or (15), depending on whether the interface is non-diffusive of diffusive, and from determination of the turbulent kinetic energy spectrum. Reference is made here to the turbulent kinetic energy spectra measured by Hannoun *et al.* (1988) in the vicinity of the interface (figure 8 of their paper). These frequency spectra show significant energy transfers at the lowest frequencies (largest scales) from the velocity component perpendicular to the interface (spectrum E_w) to the velocity component parallel to it (spectrum E_u). However, comparison of the spectra $E_w(\omega)$ and $E_u(\omega)$ shows that all spectra collapse on almost a single curve for frequencies corresponding to the wavenumber range $kl \geqslant Ri^{\frac{1}{2}}$, whose slope is apparently in good agreement with the classical $-\frac{5}{3}$ law. In the small scale range considered ($kl \geqslant (Ri\, l/e)^{\frac{1}{2}}$), turbulence is therefore nearly isotropic. These properties mean that the spectral-law distribution may be written in the classical form:

$$E_k(k) = u^2 l (kl)^{-\frac{5}{3}}. \tag{16}$$

Scaling of the spectrum with $u^2 l$ is retained because the spectra measured by Hannoun *et al.* (1988) show no modification by stratification of the spectral distribution in the range defined by (7). The spectral distribution (16), derived for a non-stratified fluid, is valid near the interface, provided it is used in that range.

A further insight into the spectral kinetic energy distribution at the interface is provided by the internal wave spectra measured by Hannoun & List (1988) and, more recently, by Fleury *et al.* (1991). In what follows, we call these spectra 'interface displacement spectra' as it is not obvious that they only account for internal wave motions. If $\zeta(t)$ denotes the time fluctuation of the middle of the interface, the interface displacement spectrum is

$$\Phi(\omega) = \langle \hat{\zeta}(\omega)\, \hat{\zeta}^*(\omega) \rangle, \tag{17}$$

$\hat{\zeta}(\omega)$ being the Fourier transform of $\zeta(t)$. The spectral measurements by Hannoun & List (1988) and Fleury *et al.* (1991) are of frequency spectra. These are Eulerian spectra because measurements are taken at a fixed position (x, y) in the tank. The frequency is related to the wavenumber by $\omega = uk$, u being the horizontal r.m.s. turbulent velocity. $\langle\ \rangle$ designates the ensemble average. The interface displacement spectrum is related to the turbulent kinetic energy spectrum $E_w(\omega, z = h)$ of the velocity component perpendicular to the interface, measured at the interface, according to

$$E_w(\omega, z = h) = \omega^2 \Phi(\omega). \tag{18}$$

The spectrum $E_w(\omega, z = h)$ is Eulerian in the horizontal coordinates (x, y) but it is a Lagrangian quantity in the vertical direction since it is obtained at the position of the interface. This spectrum differs in principle from the spectrum $E_w(\omega)$ measured by Hannoun *et al.* (1988), which is an Eulerian spectrum measured by LDA at a fixed position below the mean position of the interface. However, both quantities do not differ much for eddies with lengthscale $1/k$ such that $kl \approx (Ri\,l/e)^{\frac{1}{2}}$; at least they presumably exhibit the same power-law dependence on frequency. Actually, the vertical velocity of the interface $w_i(t)$ is related to the Eulerian vertical velocity component $w_0(t)$ at the mean position of the interface by

$$w_i(t) = w_0(t) + \zeta \frac{\partial w_0}{\partial z}(t).$$

For an eddy with wavenumber k we estimate $\partial w_0/\partial z = O(w_0 k)$ and $\zeta = O(lRi^{-1})$, implying that $(w_i - w_0)/w_0 = O(klRi^{-1})$ is a small quantity for eddies of lengthscale such that $kl \approx Ri^{\frac{1}{2}}$, when the Richardson number is sufficiently high. Interface displacement spectra provide information about the dynamics at the interface which is complementary to that obtained from the turbulent kinetic energy spectra (as measured by Hannoun *et al.* 1988). Interface displacement spectra provide an exact determination of the kinetic energy spectrum at the interface ($z = h$, see (18)), whereas the turbulent kinetic energy spectra determined by Hannoun *et al.* were measured at a position close to the interface, but somewhat below it. On the other hand, direct measurement of the turbulent kinetic energy spectra by the latter authors gives the only evidence of the isotropy of turbulence in the small-scale range. Figure 2(*a*) shows replots of an interface displacement spectrum $\Phi(\omega)$, versus frequency, measured by Fleury *et al.* (1991). In order to determine accurately the best fit of the interface displacement spectrum $\Phi(\omega)$ with a power law of the form ω^{-n}, two plots of $\omega^n\Phi(\omega)$ for the same spectrum $\Phi(\omega)$ are presented in figure 2 with $n = 3$ and $n = \frac{11}{3}$, respectively. The particular value $n = \frac{11}{3}$ is related to a turbulent kinetic energy spectrum varying like $k^{-\frac{5}{3}}$ (see (18)). The value $n = 3$ was considered by Hannoun & List (1988) in terms of Phillips' (1977) model. This point is discussed in more detail in §5, as the present analysis disagrees with certain interpretations that Hannoun & List made of their spectra. Apparently, the spectrum in figure 2(*a*) shows a better fit with a dependence in the form $\omega^{-\frac{11}{3}}$, as the spectrum $\omega^{\frac{11}{3}}\Phi(\omega)$ is almost constant in the frequency range extending between the internal wave frequency $f_i = (1/2\pi)(g'/l)^{\frac{1}{2}}$ and the cutoff frequency f_c imposed by the measurement system. Unfortunately, this frequency range is only half a decade wide.

When the spectral law (16) is introduced in (11), the rate of increase of the total potential energy becomes

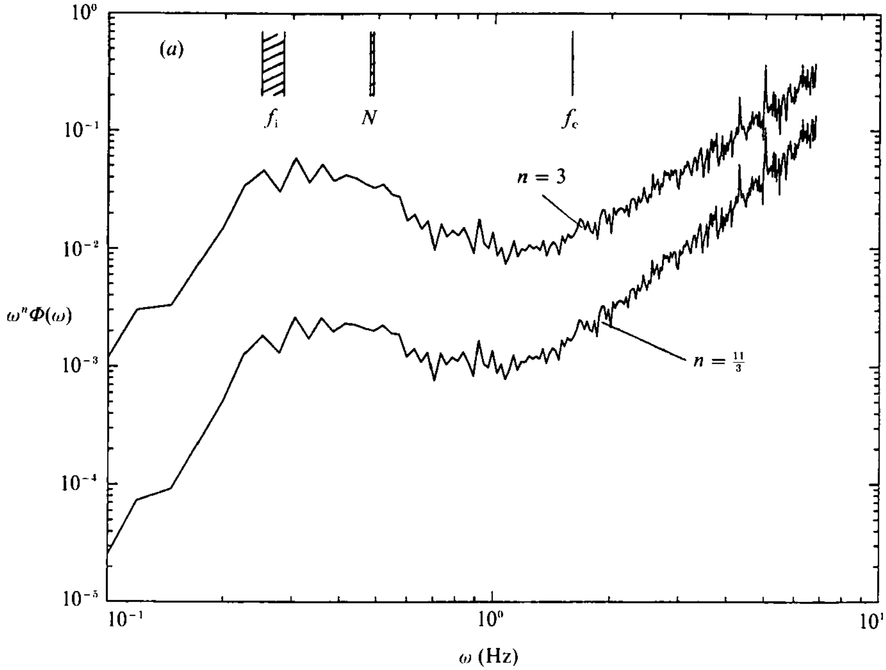
$$\frac{\mathrm{d}E_p}{\mathrm{d}t} = \Delta\rho g h u_e \approx \rho u^3 \int_{kl > (Ri\,l/e)^{\frac{1}{2}}} (kl)^{-2}\,\mathrm{d}(kl), \tag{19}$$

leading finally to the entrainment law

$$E = \frac{u_e}{u} \approx \frac{l}{h} Ri^{-1} \left(Ri \frac{l}{e}\right)^{-\frac{1}{2}}. \tag{20}$$

The two cases, respectively high and low Péclet number, now need to be examined separately. For a *non-diffusive* interface, l/e is a constant, independent of the Richardson number, (12). Hence

$$E \approx \frac{(le)^{\frac{1}{2}}}{h} Ri^{-\frac{3}{2}} \tag{21}$$



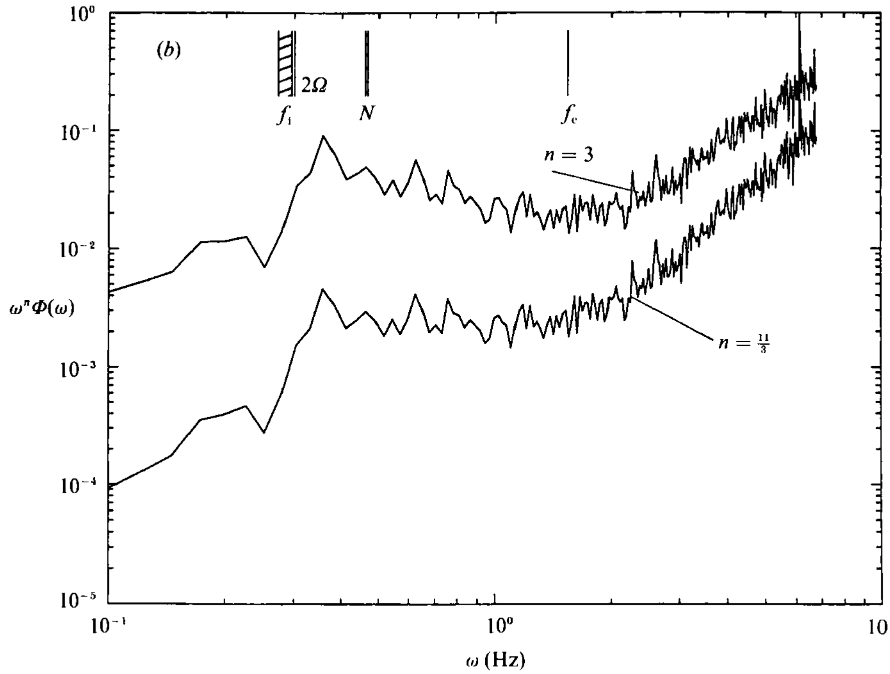


FIGURE 2. Frequency spectra $\omega^{\frac{11}{3}}\Phi(\omega)$ and $\omega^3\Phi(\omega)$ (plotted in arbitrary units), with $\Phi(\omega)$ the interface displacement frequency spectra measured by Fleury *et al.* (1991). $f_i = (1/2\pi)(g'/l)^{\frac{1}{2}}$ is the internal wave frequency, $N = (1/2\pi)(g'/e)^{\frac{1}{2}}$ the Brünt–Väisälä frequency and f_c the cutoff frequency imposed by the apparatus. (*a*) Non-rotating conditions ($\Phi(\omega)$ is shown in figure 12 of Fleury *et al.*). Richardson number range: $Ri = 39$–84. (*b*) Rotating conditions ($\Phi(\omega)$ is shown in figure 13 of Fleury *et al.*). $Ri = 23$–36.5 and $Ro = 0.18$–0.24. Ω/π is the rotation frequency.

(l/h is a constant value in the experiments referred to in the present paper). For a *diffusive* interface, the interface thickness is related to the entrainment rate E by (15). Combining this equation with (21), one obtains

$$E \approx \frac{l}{h} Ri^{-1} (Ri\, E\, Pe)^{-\frac{1}{2}}, \tag{22}$$

which then gives

$$E \approx \left(\frac{l}{h}\right)^{\frac{2}{3}} Pe^{-\frac{1}{3}} Ri^{-1}. \tag{23}$$

The dependence of entrainment on Ri in the form (21) and (23) has been confirmed by experiments. For a non-diffusive interface, the $Ri^{-\frac{3}{2}}$ dependence of the entrainment rate is supported by the experiments by Thompson & Turner (1975), Hopfinger & Toly (1976) and E & Hopfinger (1986). The case of a diffusive interface was previously investigated by Turner (1968), who established the Ri^{-1} law. Though various behaviours of the entrainment law versus the Péclet number have been proposed by different authors (Turner 1968; Hopfinger & Toly 1976), the dependence on the Péclet number has not been determined experimentally. It is worth mentioning that the entrainment law (23) obtained for diffusive conditions implies the following dependence for the interface thickness:

$$\frac{e}{l} \approx \left(\frac{h}{l}\right)^{\frac{2}{3}} Pe^{-\frac{2}{3}} Ri. \tag{24}$$

The increase of the interface thickness with increasing Richardson number is a remarkable result since the opposite trend is observed when the Péclet number is very high (Hannoun & List 1988; E & Hopfinger 1986). Crapper & Linden (1974) actually observed a tendency of the interface thickness to increase with increasing Richardson number for moderate Péclet number ($Pe = 155$ in figure 4 of their paper), though their data did not permit a quantitative determination of this dependence.

4. Application to rotating turbulent flows

The rate of increase of the potential energy is still given by (11). When a mean rotation is superimposed, its effect on the turbulent kinetic energy spectrum $E_k(k)$ at the interface and on the criterion for instability, (4), (which determines the domain of integration) needs to be determined in more detail.

Unfortunately, there is no turbulent kinetic energy spectrum available in rotating conditions similar to the spectra measured by Hannoun *et al.* (1988) in the non-rotating case. Nevertheless, interface displacement spectra $\Phi(\omega)$ have been measured in various rotating conditions by Fleury *et al.* (1991), and compared by them to interface displacement spectra measured in the absence of rotation. This comparison reveals several important features. In the experiments carried out by Fleury *et al.* the internal wave frequency $(g'/l)^{\frac{1}{2}}$ and the Coriolis frequency f are of the same order of magnitude. On the one hand, the interface displacement spectra obtained in rotating conditions show the emergence of important oscillations of the interface at frequencies of the order of magnitude of the turbulence frequency u/l, which is much lower than $(g'/l)^{\frac{1}{2}}$ and f. Oscillations at these low frequencies are one order of magnitude weaker in non-rotating conditions. In the framework of the present theory, the large-scale motions associated with the low-frequency oscillations do not take part in the mixing. Fleury *et al.* (1991) relate their existence to inertial waves that transmit kinetic energy across the interface into the non-mixed layer. On the other hand, the interface displacement spectrum is not modified by rotation for frequencies above $(g'/l)^{\frac{1}{2}}$ and f. Modifications of the interface displacement spectrum caused by rotation only affect frequencies smaller than the frequency of rotation and the internal wave frequency $(g'/l)^{\frac{1}{2}}$. The definition of the various characteristic frequencies of the flow and a detailed discussion of the shape of the interface displacement spectra in the various frequency ranges are contained in §6 of Fleury *et al.* (1991). In the present paper we focus on the high-frequency range corresponding to eddies active in the mixing process; rotation has no effect on interface displacements at these scales. Figure 2(*b*) shows an interface displacement spectrum $\Phi(\omega)$ measured in rotating conditions. Actually the spectra $\omega^3\Phi(\omega)$ and $\omega^{\frac{11}{3}}\Phi(\omega)$ are plotted to distinguish which of the power laws ω^{-3} and $\omega^{-\frac{11}{3}}$ is more representative of the measured spectra. The best agreement is $\Phi(\omega) \propto \omega^{-\frac{11}{3}}$ because the spectrum $\omega^{\frac{11}{3}}\Phi(\omega)$ is almost constant over half a decade for $(1/2\pi)\,(g'/l)^{\frac{1}{2}} < \omega$ and $f/\pi < \omega < f_c$ (f_c is the cutoff frequency imposed by the measurement technique). Equation (18) relates the interface displacement spectrum to the turbulent kinetic energy spectrum $E_w(\omega, z = h)$ at the interface of the velocity component perpendicular to it. The turbulent kinetic energy spectrum is not modified by rotation for $f/\pi < \omega < f_c$ and $(1/2\pi)\,(g'/l)^{\frac{1}{2}} < \omega$. The spectra are Eulerian in the (x, y)-coordinates, so that the relationship between frequency and wavenumber is linear ($\omega = uk$). In the range defined above, the turbulence kinetic energy spectrum is expressed versus wavenumber as

$$E_k(k) = u^2 l (kl)^{-\frac{5}{3}}, \tag{25}$$

provided turbulence remains isotropic in this frequency range. This property cannot be proved by the experimental results available, but is likely to be true. It is physically reasonable that rotation should not modify turbulence at frequencies higher than the rotation frequency. This property is verified by the interface displacement spectra. There is no reason why turbulence should become non-isotropic in this frequency range, whereas it is isotropic in this range for non-rotating conditions.

Turning our attention to the stability criterion defining the integration domain in (11), it is to be expected that this criterion should hold when the Eulerian frequency $\omega_E \approx uk$ (equation (6)) is higher than the Coriolis frequency f for all eddies contributing to the mixing of the density profile. This implies that the entrainment law obtained without rotation, (21), (non-diffusive interface) still applies for a rotating system when

$$\frac{u}{l}\left(Ri\,\frac{l}{e}\right)^{\frac{1}{2}} \geqslant f, \tag{26}$$

implying

$$Ri^{\frac{1}{2}} Ro \left(\frac{l}{e}\right)^{\frac{1}{2}} \geqslant 1. \tag{27}$$

When the reverse inequality is satisfied, in contrast, it is to be expected that turbulent eddies with wavenumbers in the range

$$\left(Ri\,\frac{l}{e}\right)^{\frac{1}{2}} \leqslant kl \leqslant Ro^{-1} \tag{28}$$

should be affected by rotation, because the corresponding frequencies are lower than the rotation frequency f/π. It is well known that rotation usually acts as a stabilizing effect. This property is satisfied by the Kelvin–Helmholtz instability (Chandrasekhar 1961; Huppert 1968). Assuming that rotation will cause maximum stabilization, it will totally inhibit mixing by turbulent eddies in the range defined by (28), whereas these eddies would mix the density profile if rotation were eliminated. In spite of its crudeness, this assumption is consistent with the general trend of rotation stabilizing instabilities. It will be verified *a posteriori* that such a strong assumption leads to an entrainment law in agreement with the experimental results by Fleury *et al.* (1991).

Under the condition

$$\left(Ri\frac{l}{e}\right)^{\frac{1}{2}} Ro \leqslant 1, \tag{29}$$

the rate of increase of the potential energy (11) is modified to

$$\frac{\mathrm{d}E_{\mathrm{p}}}{\mathrm{d}t} = \Delta\rho g h u_{\mathrm{e}} \approx \rho \int_{kl > Ro^{-1}} E_{\mathrm{k}}(k)\,(E_{\mathrm{k}}(k)\,k)^{\frac{1}{2}}\,\mathrm{d}k, \tag{30}$$

where only the integration range has been changed to account for the stabilizing effect of rotation. With the spectral energy distribution (25), one obtains after integration

$$\Delta\rho g h u_{\mathrm{e}} \approx \rho u^3 Ro, \tag{31}$$

leading finally to the entrainment law

$$E \approx \frac{l}{h} Ro\, Ri^{-1}. \tag{32}$$

The dependence of the entrainment rate on the Rossby and Richardson numbers has actually been demonstrated in experiments by Fleury *et al.* (1991).

5. Discussion

In the present paper three entrainment laws have been established theoretically for mixing across an interface of finite thickness by a turbulent flow having no mean shear. These three laws are valid for well-defined asymptotic regimes. The results are summarized in table 1. This table also refers to the experiments, which confirm the entrainment laws obtained theoretically.

In rotating conditions, the entrainment law $E \approx (l/h) Ro\, Ri^{-1}$ is clearly observed in experiments by Fleury *et al.* (1991). Unfortunately, sufficiently large Richardson numbers could not be achieved in these experiments to verify whether the non-rotating entrainment law $E \approx (le/h^2)^{\frac{1}{2}} Ri^{-\frac{3}{2}}$ is recovered for high values of the parameter $Ri^{\frac{1}{2}} Ro$. Data obtained in this range tend to confirm this behaviour, but are to scarce to demonstrate any non-dimensional entrainment law. However, a behaviour that differed from that predicted by the model would be surprising, as it would imply that rotation modifies the structure of turbulence even when the Rossby number is asymptotically large. We regard the prediction of two regimes depending on the value of $Ri^{\frac{1}{2}} Ro$ as an additional advantage of the model.

Two models of entrainment, not accounting for the effect of rotation, have been proposed by Linden (1973): $E \propto Ri^{-\frac{3}{2}}$ and Long (1978): $E \propto Ri^{-\frac{7}{4}}$ for a non-diffusive interface. The entrainment law proposed by Linden (without rotation) is confirmed, but the differences between the present approach and Linden's need to be emphasized. Linden associates the mixing with large-scale eddies impinging on the interface and carrying the non-turbulent fluid into the mixed layer during recoil. The

Type of interface	Domain of validity	Entrainment law $E = u_e/u$	Experimental references
		Without rotation	
Non-diffusive $Pe = ul/\kappa \to \infty$	$Ri \gg 1$	$E \propto (le)^{\frac{1}{2}}/h\, Ri^{-\frac{3}{2}}$	Thompson & Turner (1975); Hopfinger & Toly (1976); E & Hopfinger (1986)
Diffusive Pe moderate	$Ri \gg 1$	$E \propto (l/h)^{\frac{2}{3}} Pe^{-\frac{1}{3}} Ri^{-1}$	Turner (1968)
		With rotation	
Non-diffusive $Pe = ul/\kappa \to \infty$	$Ri \gg 1$, $(l/e)^{\frac{1}{2}} Ro\, Ri^{\frac{1}{2}} \geqslant 1$	$E \propto (le)^{\frac{1}{2}}/h\, Ri^{-\frac{3}{2}}$	No effect of rotation
	$Ri \gg 1$, $(l/e)^{\frac{1}{2}} Ro\, Ri^{\frac{1}{2}} \leqslant 1$	$E \propto l/h\, Ro\, Ri^{-1}$	Fleury *et al.* (1991)

TABLE 1. Entrainment laws

estimate of the potential energy increase made here, (9), is dimensionally similar to Linden's, but the mixing is a small-scale process whereas mixing is related by Linden to the dynamics of large-scale eddies. The estimates of the available kinetic energy and of the time- and lengthscales are therefore very different in the two models. Moreover, Linden considers the interaction of a single large-scale eddy with the interface. When such a model is applied to fully developed turbulence at the interface, the mixing events of the type described by Linden presumably arise intermittently. In Linden's model, the arrival timescale of eddies at the interface is the turbulent turnover time, whereas the typical duration of the interaction of the eddy with the interface is the timescale of long internal waves. Intermittency is accounted by the fact that the latter timescale is much smaller than the former. The model proposed by Long (1978) presents some similarities with the new approach developed here. Mixing is considered by Long as being caused by the breaking of internal waves at the interface. His model assumes, as does the present one, that mixing is a small-scale process governed by an instability whose criterion is of the form given by (4). One difference is that intermittency is necessary in Long's model, while it is neglected here. A disadvantage of Long's model is its complexity. It is difficult to understand all the steps of his analysis. Some results seem doubtful, like the ratio of the r.m.s. velocity component perpendicular to the interface to the r.m.s. velocity component parallel to the interface in the vicinity of the latter. Long's model predicts $w/u \propto Ri^{-\frac{1}{4}}$, whereas the measurements by Hannoun & List (1988), confirmed by Fleury (1988), show a $w/u \propto Ri^{-\frac{1}{2}}$ dependence.

The simplicity of the present approach and its ability to predict correctly the entrainment laws for three different regimes argue in its favour. A key point is the use of two different timescales: the criterion for instability is expressed using the Eulerian timescale, and that for the mixing itself by the Lagrangian timescale (see §2). Introducing two different frequencies in our dimensional scaling, (9), obviously increases the number of possible forms of entrainment laws that may be derived. Assuming that the frequency $\omega(k)$ in the integrand of (9) is in the form $uk/(kl)^p$ whereas the frequency defining the integral limit is $uk/(kl)^r$, it is quite simple to obtain the following entrainment laws:

(i) $E \propto Ri^{-(1+s)}$ without rotation and for $Pe \gg 1$;

(ii) $E \propto Ri^{-1} Pe^{-s/(s+1)}$ without rotation and for small or moderate Péclet number;
(iii) $E \propto Ri^{-1} Ro^{2s}$ with rotation and for $Pe \gg 1$.

The number s is $s = (p+\frac{2}{3})/2(1-r)$. These simple relations† point out how important the dependence of entrainment laws on the Péclet and Rossby numbers is for a test of the theory. The $Ri^{-\frac{3}{2}}$ law for a non-diffusive interface without rotation and the Ro dependence in the presence of rotation (non-diffusive interface) are consistent, implying that $p+r=\frac{1}{3}$. The predicted Pe dependence without rotation when the interface is diffusive is therefore like $Pe^{-\frac{1}{3}}$. Unfortunately, the Pe dependence has not been measured. Obviously, many couples (p, r) verify the condition $p+r=\frac{1}{3}$, among which are the values $(\frac{1}{3}, 0)$ used in the present model. Introducing the Eulerian frequency inside the integrand and the Lagrangian frequency in the integral limit would give the same results. Nevertheless not any values for p and r can be accepted because the defined frequencies must have physical meaning, as the Eulerian and Lagrangian frequencies have. A linear model would involve using the Eulerian frequency only. It is more consistent to introduce the Eulerian frequency in the integral limit as linear stability should determine the range of eddies that will be able to mix. Using the Lagrangian frequency as a timescale for mixing implies that mixing is a nonlinear process. A second nonlinear feature of the model is the incorporation of the turbulent kinetic energy spectrum, which played no role in the models proposed by Linden and Long. It seems physically reasonable that the turbulent kinetic energy distribution should enter in some way into the entrainment law. Owing to the decay of the kinetic energy spectrum, integration over the wavenumber range of eddies contributing to mixing leads to a very simple result.

The present work has been largely influenced by the very important experimental results obtained by Hannoun *et al.* (1988) and by Hannoun & List (1988). However, Hannoun & List's interpretation of their internal wave spectra in terms of Phillips' (1977) theory of mixing seems incorrect. This theory predicts that the interface displacement spectrum will correspond to the marginal state of stability for all internal waves. It then leads to an internal wave decay in the form

$$\Phi(\omega) \propto \omega^{-3}. \tag{33}$$

The interface displacement spectrum is related by $E_w(\omega) \propto \omega^2 \Phi(\omega)$, (18), to the turbulent kinetic energy spectrum $E_w(\omega, z=h)$ of the velocity component perpendicular to the interface. Energy spectra, measured by Hannoun *et al.* (1988), show a dependence in the form $E_w(\omega) \propto \omega^{-\frac{5}{3}}$ (even in the close neighbourhood of the interface) that contradicts the $\Phi(\omega) \propto \omega^{-3}$ dependence used by Hannoun & List (1988) to interpret their interface displacement spectra. The turbulent kinetic energy spectra measured by Hannoun *et al.* ($E_w(\omega) \propto \omega^{-\frac{5}{3}}$) imply an interface displacement spectrum in the form $\Phi(\omega) \propto \omega^{-\frac{11}{3}}$. It is in fact difficult to distinguish whether the interface displacement spectra obtained by Hannoun & List (figure 13) decay like $\omega^{-\frac{11}{3}}$ or ω^{-3}. The consistency between turbulent energy spectra and interface displacement spectra would indicate giving preference to $\Phi(\omega) \propto \omega^{-\frac{11}{3}}$, in agreement with the results of Fleury *et al.* (1991). An important physical consequence is that interface oscillations are not in a state of marginal stability as proposed by Phillips (1977). Instead, their distribution is imposed by the distribution of energy below the interface. One reason why Phillip's ideas seem inapplicable in the experiments referred to in the present paper is that the ratio of the Brunt–Väisälä frequency $N_m = (g'/e)^{\frac{1}{2}}$ to the large-scale internal wave frequency $n_a = (g'/l)^{\frac{1}{2}}$ is usually low. Fleury *et al.* (1991) measured neither the interface thickness nor the integral

† We are indebted to one of the referees for this argument.

lengthscale of turbulence. Using the measurements of the interface thickness obtained by E & Hopfinger (1986) and the integral lengthscales measured by Mory & Hopfinger (1985) they estimated the ratio N_m/n_a to be of the order of 2 to 3. The condition $N_m/n_a \gg 1$ is also not be reached in Hannoun & List's (1988) experiments. For an overall Richardson numbers in the 24–100 range, figure 10 of their paper indicates a ratio of N_m/n_a in the range 2.8–5.5. Considering all the points mentioned above, we therefore believe that the data by Hannoun & List are more satisfactorily interpreted by the present model than by Phillips' model. Nevertheless, the determination of the interface thickness is still a subject of controversy. Interface thicknesses measured by E & Hopfinger (1986) and by Hannoun & List (1988) are consistent at Richardson numbers of the order of 20–30, but much smaller in the experiments by Hannoun & List than in E & Hopfinger's experiments when the Richardson number is higher ($e/l = 0.03$ as compared to $e/l = 0.22$ for $Ri = 100$). Moreover Hannoun & List believed that their measurements were overestimated from a lack of resolution of their laser-induced fluorescence technique. Effects of averaging concentration profiles measured by a conductivity probe, as shown by Hannoun & List in their figure 2 cannot explain the thicker interface thickness obtained by E & Hopfinger. The explanation of this discrepancy remains an open question. However, as mentioned before, the only hypothesis used in our model is that e/l is independent of the Richardson number for sufficiently large values. It can be small or of the order of one.

The role of dissipation must finally be considered to justify the assumption that the small-scale eddies active in the mixing process $kl > (Ri\, l/e)^{\frac{1}{2}} \approx Ri^{\frac{1}{2}}$ are sufficiently large compared to the Kolmogorov lengthscale. The wavenumber associated with the Kolmogorov lengthscale is $k_D\, l = (ul/\nu)^{\frac{3}{4}}$. The condition required is therefore

$$Ri^{\frac{1}{2}} \ll (ul/\nu)^{\frac{3}{4}}. \tag{34}$$

This condition is satisfied in Fleury *et al.*'s (1991) experiments. Without rotation (figure 2*a*) $Ri^{\frac{1}{2}} \approx 6$–$9$ and $(ul/\nu)^{\frac{3}{4}} \approx 144$, while $Ri^{\frac{1}{2}} \approx 5$ and $(ul/\nu)^{\frac{3}{4}} \approx 112$ with rotation (figure 2*b*). The factor of ten between the size of eddies considered for mixing and the Kolmogorov lengthscale cannot, however, be considered large. Studies by Gibson (1981) and Stillinger, Helland & Van Atta (1983) have shown that dissipation is efficient up to a lengthscale about ten times larger than the Kolmogorov lengthscale. Nevertheless, these authors considered flow with a linear mean density profile in which turbulence is produced by a mean flow passing through a grid perpendicular to the stratification planes. It is by no means obvious that the constant of proportionality found in this configuration is the same in the experiments with two-layer stratification where turbulence is produced by the oscillation of a grid parallel to the stratification planes. The observation by Fleury *et al.* and by Hannoun & List of power-law decays of the interface displacement spectra in the range considered for mixing indicates that eddies are presumably inertial, though the Reynolds number is not as high as might be desired.

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